

WP Metric Geometry Quick View

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Let \mathcal{T} be the Teichmüller space for marked genus g , n punctured Riemann surfaces R of negative Euler characteristic. By Uniformization a conformal structure determines a unique complete compatible hyperbolic metric ds^2 . \mathcal{T} is a complex manifold of $\dim_{\mathbb{C}} = 3g - 3 + n$ with the cotangent space at R represented by $Q(R)$, the space of holomorphic quadratic differentials on R with at most simple poles at the punctures. The Weil-Petersson Hermitian cometric is

$$\langle \varphi, \psi \rangle = \int_R \varphi \bar{\psi} (ds^2)^{-1}.$$

The WP metric is invariant under the action of the *mapping class group* MCG, [Iva02]. The metric is Kähler, non complete with negative sectional curvature with $\sup_{\mathcal{T}} = 0$ (except for $\dim \mathcal{T} = 1$) and $\inf_{\mathcal{T}} = -\infty$. *In practice and experience the WP geometry of \mathcal{T} corresponds to the hyperbolic geometry of surfaces.*

The augmented Teichmüller space, [Abi77, Ber74]. The augmented Teichmüller space $\bar{\mathcal{T}}$ is a MCG invariant bordification (a partial compactification) following the approach for Baily-Borel bordifications. ($\bar{\mathcal{T}}$ is an analog of the $SL(2; \mathbb{Z})$ invariant bordification \mathbb{Q} for \mathbb{H} provided with the Satake horocycle topology.) The bordification is also described as the Chabauty topology closure of the faithful cofinite representations of $\pi_1(R)$ into $PSL(2; \mathbb{R})$. $\bar{\mathcal{T}}/MCG$ is homeomorphic to the Deligne-Mumford stable curve compactification of the moduli space of Riemann surfaces. The elements of $\bar{\mathcal{T}} - \mathcal{T}$ are marked degenerate hyperbolic structures for which certain simple non peripheral closed curves are *represented* by pairs of cusps.

CAT(0) geometry. Important is that $\bar{\mathcal{T}}$ is WP complete with $(\bar{\mathcal{T}}, d_{WP})$ a CAT(0) metric space (a generalized complete, simply connected, non positively curved space.) The large-scale geometry is described by $\bar{\mathcal{T}}$ being quasi isometric to $\mathcal{PG}(R)$ the *pants graph* of R with unit-length edge metric, [Bro03]. $\bar{\mathcal{T}}$ is a stratified space with each open strata characterized as the union of all geodesics (distance realizing paths) containing a given point as an interior point. $\bar{\mathcal{T}}$ itself is characterized as the closed WP convex hull of the maximally degenerate hyperbolic structures (the unions of thrice punctured spheres.) An application of the CAT(0) geometry and the important rigidity of the *complex of curves* $\mathcal{C}(R)$, [Iva02, Chap. 3] is that the isometry group Isom_{WP} coincides with MCG.

Geodesic-length functions. Associated to each non trivial, non peripheral free homotopy class on R is the length of the unique ds^2 geodesic in the homotopy class. Geodesic-lengths $\ell_{\alpha}(R)$ on \mathcal{T} are elements of the WP geometry. The Fenchel-Nielsen twist (right earthquake) deformation about α is described by $2t_{\alpha} = J \text{grad } \ell_{\alpha}$ for J the Teichmüller almost complex structure. The WP Hermitian pairing is described in terms of the hyperbolic trigonometry for $R = \mathbb{H}/\Gamma$ with

$$\langle \text{grad } \ell_{\alpha}, J \text{grad } \ell_{\beta} \rangle = 4\omega_{WP}(t_{\alpha}, t_{\beta}) = -2 \sum_{p \in \alpha \cap \beta} \cos \theta_p$$

for the geodesics α, β and intersection angles θ_* on R and in upper half plane

$$\langle \text{grad } \ell_\alpha, \text{grad } \ell_\beta \rangle = \frac{2}{\pi} (\ell_\alpha \delta_{\alpha\beta} + \sum'_{\langle A \rangle \setminus \Gamma / \langle B \rangle} (u \log \frac{u+1}{u-1} - 2))$$

for the Kronecker delta δ_* , where for $C \in \langle A \rangle \setminus \Gamma / \langle B \rangle$ then $u = u(\tilde{\alpha}, C(\tilde{\beta}))$ is the cosine of the intersection angle if the lifts $\tilde{\alpha}$ and $C(\tilde{\beta})$ intersect and is otherwise $\cosh d(\tilde{\alpha}, C(\tilde{\beta}))$.

WP convexity and curvature. Information on geodesics is provided by the strict convexity of geodesic-lengths (more generally convexity of the total length of a *measured geodesic lamination*) along WP geodesics (square roots are also convex.) For surfaces with cusps the distances along geodesics between horocycles are also strictly convex. Geodesic-length sublevel sets are convex. On \mathcal{T} the WP Levi-Civita connection D is described for small geodesic-length, $\lambda = \text{grad } \ell_\alpha^{1/2}$ and tangent U by

$$D_U \lambda_\alpha = 3\ell_\alpha^{-1/2} \langle J\lambda_\alpha, U \rangle J\lambda_\alpha + O(\ell_\alpha^{3/2} \|U\|).$$

The remainder term constant is uniform for bounded geodesic-length. Bounds for the gradient and Hessian of geodesic-length give rise to applications. The WP curvature of the span $\{t_\alpha, Jt_\alpha\}$ is $O(-\ell_\alpha^{-1})$. Similarly for a pair of deformations supported on different components of $R - \{\text{short geodesics}\}$ the curvature of the corresponding 2-plane is $O(-\text{short separating lengths})$.

Fenchel-Nielsen coordinates. Marked hyperbolic pairs of pants are determined by their three boundary geodesic-lengths in \mathbb{R}_+ . The FN twist-length coordinates $(\ell_j, \tau_j)_{j=1}^{3g-3+n}$ for assembling hyperbolic pants provide global coordinates for \mathcal{T} with expansions

$$\omega_{WP} = \frac{1}{2} \sum d\ell_j \wedge d\tau_j$$

and on the Bers region $\{\ell_j < c_0\}$ the expansions

$$\langle \cdot, \cdot \rangle \asymp \sum (d\ell_j^{1/2})^2 + (d\ell_j^{1/2} \circ J)^2 \asymp \sum \text{Hess } \ell_j$$

with comparability uniform in terms of c_0 . At the maximally degenerate structure the WP metric has the expansions

$$\begin{aligned} \langle \cdot, \cdot \rangle &= 2\pi \sum (d\ell_j^{1/2})^2 + (d\ell_j^{1/2} \circ J)^2 + O(\sum \ell_j^3 \langle \cdot, \cdot \rangle) \\ &= \frac{\pi}{6} \sum \frac{\text{Hess } \ell_j^2}{\ell_j} + O(\sum \ell_j^2 \langle \cdot, \cdot \rangle). \end{aligned}$$

Alexandrov tangent cone. For a pair of geodesics (distance realizing curves) from a common point in a $CAT(0)$ metric space there is a well-defined initial angle, [BH99, Chap. II.3]. A tangent cone is defined in terms of initial angle. At a point p of a strata $\mathcal{T}(\sigma) \subset \overline{\mathcal{T}} - \mathcal{T}$, $\sigma \in \mathcal{C}(R)$, the WP Alexandrov tangent cone AC_p is isometric to a product of a Euclidean orthant and the

tangent space $\mathbf{TT}(\sigma)$ with WP pairing. The isometry is given in terms of initial derivatives of geodesic-length functions. The dimension of the Euclidean orthant is the count of geodesic-lengths trivial on $\mathcal{T}(\sigma)$ (the count of nodes.) A property of $\overline{\mathcal{T}}$ is that a WP geodesic tangential to $\mathcal{T}(\sigma)$ at p actually lies in $\mathcal{T}(\sigma)$. Applications include that geodesics do not *refract* at the bordification and that the angle of incidence and reflection coincide for certain limits of degenerating geodesics. A further application is for combinatorial harmonic maps. Certain groups acting on Euclidean buildings and group extensions acting on Cayley graphs satisfying a Poincaré type inequality for links of points will have a global fixed point for an action on $\overline{\mathcal{T}}$.

All important attributions and references are provided in the introductions of [Wol03, Wol07].

References

- [Abi77] William Abikoff. Degenerating families of Riemann surfaces. *Ann. of Math. (2)*, 105(1):29–44, 1977.
- [Ber74] Lipman Bers. Spaces of degenerating Riemann surfaces. In *Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973)*, pages 43–55. Ann. of Math. Studies, No. 79. Princeton Univ. Press, Princeton, N.J., 1974.
- [BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*. Springer-Verlag, Berlin, 1999.
- [Bro03] Jeffrey F. Brock. The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores. *J. Amer. Math. Soc.*, 16(3):495–535 (electronic), 2003.
- [Iva02] Nikolai V. Ivanov. Mapping class groups. In *Handbook of geometric topology*, pages 523–633. North-Holland, Amsterdam, 2002.
- [Wol03] Scott A. Wolpert. Geometry of the Weil-Petersson completion of Teichmüller space. In *Surveys in Differential Geometry VIII: Papers in Honor of Calabi, Lawson, Siu and Uhlenbeck*, pages 357–393. Intl. Press, Cambridge, MA, 2003. on Arxiv: math/0502528.
- [Wol07] Scott A. Wolpert. Behavior of geodesic-length functions on Teichmüller space. preprint on Arxiv: math/0701556, 2007.